Fast Evaluation of Modal Coupling Coefficients of Waveguide Step Discontinuities

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Abstract— **This letter presents a fast method for the determination of a large number of coupling coefficients of a step discontinuity between a rectangular/circular waveguide and a smaller one of arbitrary shape. The mode spectrum of the small waveguide is calculated by a fast algorithm—presented some years ago—and the coupling coefficients are obtained at almost no extra cost by post-processing some of the matrices already used for finding the modes. Tens of modes and hundreds of coefficients are calculated in a few seconds on an ordinary workstation.**

I. Introduction

Many modern simulation tools for the design of complicated waveguide junctions, such as filters and diplexers, are based on the modeling of the whole circuit as a cascade of step discontinuities between pairs of waveguides. Each discontinuity is characterized by a generalized scattering or admittance matrix [1]–[3], whose computation requires to calculate a large number of coupling integrals between the modal fields of the two waveguides. This is no problem when the mode spectrum of both waveguides is known analytically, e.g., when considering junctions between rectangular and/or circular waveguides. In many cases, however, it is necessary to study the junction between a rectangular or circular waveguide Ω and a smaller waveguide S of arbitrary cross section (see Fig. 1). Here, two problems arise: 1) the computation of the mode spectrum of S and 2) the evaluation of a large number of modal fields of S in many points of its cross section to compute the coupling integrals numerically. Among the many known methods to find the mode spectrum of an arbitrarily shaped waveguide, the algorithm described in [4] may help in solving both problems. In fact, this method not only allows one to compute a large number of modes in a short time, but it also has the peculiarity of defining the modes of the small waveguide in an enlarged domain of rectangular or circular shape, thus reproducing the geometry of Fig. 1. This fact turns out to be very convenient, since the coupling coefficients can be calculated straightforwardly—without cumbersome field integrations—from the entries of some of the matrices already used in the mode determination or of other matrices that are easily computed. This possibility, not considered in [4], is described in this letter.

II. Expressions of the Coupling Coefficients

Denoting by $\vec{\mathcal{E}}'_p$ ($\vec{\mathcal{E}}''_p$) the normalized electric modal vector of the pth TE (TM) mode in Ω, and by $\vec{e}'_q$ ($\vec{e}''_q$) the normalized electric modal vector of the qth TE (TM) mode in S, the coupling coefficients to be evaluated are

$$I_{pq}^{\mathrm{TE,\,TE}} = \int_S \vec{\mathcal{E}}'_p \cdot \vec{e}'_q \, ds \; ; \; I_{pq}^{\mathrm{TM,\,TM}} = \int_S \vec{\mathcal{E}}''_p \cdot \vec{e}''_q \, ds$$
$$I_{pq}^{\mathrm{TM,\,TE}} = \int_S \vec{\mathcal{E}}''_p \cdot \vec{e}'_q \, ds \; ; \; I_{pq}^{\mathrm{TE,\,TM}} = \int_S \vec{\mathcal{E}}'_p \cdot \vec{e}''_q \, ds.$$

The method described in [4] yields the cutoff wavenumber and the electric field for the first TE and TM modes of S by solving two linear matrix eigenvalue problems. Denoted by κ'_q the cutoff wavenumber of the qth TE mode—resulting from the qth eigenvalue of (11) in [4]—and by $\{\{b'_n\}; \{a'_m\}\}$ the corresponding eigenvector, partitioned as described in [4], the modal vector $\vec{e}'_q$ is given by

$$\vec{e}'_q(\vec{r}) = \kappa'_q \sum_{m=1}^{M'} \frac{a'_m}{k'^2_m} \vec{\mathcal{E}}'_m(\vec{r}) + \frac{1}{\kappa'_q} \sum_{n=1}^{N'} b'_n \nabla_T \int_\sigma g(\vec{r}, \vec{s}) \frac{\partial w_n(\ell)}{\partial \ell} \, d\ell + \kappa'_q \sum_{n=1}^{N'} b'_n \int_\sigma \overleftrightarrow{G}_{st}(\vec{r}, \vec{s}) \cdot \vec{t}(\ell) \, w_n(\ell) \, d\ell \quad (1)$$

where $\vec{r}$ is the position vector; σ is the boundary of S; $\vec{s} = \vec{s}(\ell)$ denotes a generic point on σ (ℓ represents an abscissa taken on σ); $\vec{t}(\ell)$ is the unit vector tangent to σ; ∇_T is the two-dimensional (2-D) nabla operator; $\{w_n\}$ is a set of N' basis functions defined on σ; g and $\overleftrightarrow{G}_{st}$ are the scalar and dyadic Green functions defined in [4, eqs. (2)–(5)]; k'_m is the cutoff wavenumber of the mth TE mode in Ω. The first term on the r.h.s. of (1) represents a modal expansion involving the first M' TE modes of Ω. Equation (1) is obtained by imposing the normalizing condition on S to the field expressed by (13) in [4]; it assumes that the eigenvectors are normalized as specified in [5, eqs. (A2) and (A3)], in the context of an equivalent eigenvalue problem.

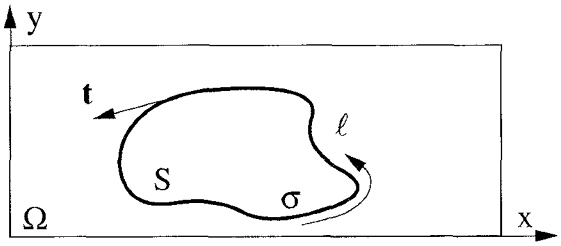

Fig. 1. The geometry of the problem. The large waveguide Ω can be either rectangular or circular.

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The modal field of the qth TM mode is calculated as

$$\vec{e}''_q = -\frac{\nabla_T \phi_q}{\kappa''_q} \tag{2}$$

where κ''_q is its the cutoff wavenumber—deduced from the qth eigenvalue of eq. (15) in [4]—and ϕ_q is the corresponding scalar potential. Function ϕ_q, in turn, can be obtained from [4, eq. (19)], after normalization on S

$$\phi_q(\vec{r}) = \kappa''_q \left[\sum_{m=1}^{M''} \frac{a''_m}{k''^2_m} \varphi_m(\vec{r}) + \sum_{n=1}^{N''} b''_n \int_\sigma g(\vec{r}, \vec{s}) w_n(\ell)\, d\ell \right] \tag{3}$$

where φ_m is the mth normalized eigenfunction of the homogeneous Helmoltz equation in Ω with Dirichlet's boundary conditions, k''_m is the corresponding eigenvalue, $\{a''_m\}$ are the entries of the qth eigenvector of eq. (15) in [4], and $\{b''_n\}$ are obtained from the a''_m by eq. (18) in [4]. Also in (3) the first term on the r.h.s. represents a modal expansion involving the first M'' φ-eigenfunctions. The normalizing condition of $\{a''_m\}$ is deduced in [6], using a somewhat different notation. With the symbols used here this condition reads $\sum_{m=1}^{M''} |a''_m|^2 = \kappa''^2_q$.

To evaluate the coupling coefficients let us consider the eigenfunction expansions of g [7] and $\overleftrightarrow{G}_{st}$ [4]

$$g = \sum_{i=1}^{\infty} \frac{\varphi_i(\vec{r})\varphi_i(\vec{r})}{k''^2_i}$$

$$\overleftrightarrow{G}_{st} = \sum_{i=1}^{\infty} \frac{\vec{\mathcal{E}}'_i(\vec{r})\vec{\mathcal{E}}'_i(\vec{r})}{k'^2_i}. \tag{4}$$

From (1) and (2), substituting (3) and (4) and recalling that $\vec{\mathcal{E}}''_i = -\nabla_T \varphi_i / k''_i$, the following expressions are obtained

$$\vec{e}'_q = \kappa'_q \left[\sum_{n=1}^{N'} b'_n \sum_{i=1}^{\infty} \frac{\vec{\mathcal{E}}'_i}{k'^2_i} \int_\sigma \vec{\mathcal{E}}'_i \cdot \vec{t} w_n \, d\ell + \sum_{m=1}^{M'} \frac{a'_m}{k'^2_m} \vec{\mathcal{E}}'_m \right] - \frac{1}{\kappa'_q} \sum_{n=1}^{N'} b'_n \sum_{i=1}^{\infty} \frac{\vec{\mathcal{E}}''_i}{k''_i} \int_\sigma \varphi_i \frac{\partial w_n}{\partial \ell} \, d\ell \tag{5}$$

$$\vec{e}''_q = \sum_{n=1}^{N''} b''_n \sum_{i=1}^{\infty} \frac{\vec{\mathcal{E}}''_i}{k''_i} \int_\sigma \varphi_i w_n \, d\ell + \sum_{m=1}^{M''} \frac{a''_m}{k''_m} \vec{\mathcal{E}}''_m. \tag{6}$$

Moreover, it is observed that, as pointed out in [4], (1) and (3)—and consequently (5) and (6)—give rise to field values that are zero when $\vec{r}$ is taken in the region $\Omega - S$. Therefore, the coupling coefficients can be evaluated carrying out the integration in the whole domain Ω, where it is possible to take advantage from the orthonormality of vectors $\vec{\mathcal{E}}'_p$ and $\vec{\mathcal{E}}''_p$. It is finally obtained

$$I^{\mathrm{TE,TE}}_{pq} = \int_\Omega \vec{\mathcal{E}}'_p \cdot \vec{e}'_q \, ds = \kappa'_q \left[\sum_{n=1}^{N'} R_{np} b'_n + \frac{a'_p}{k'^2_p} \right] \tag{7}$$

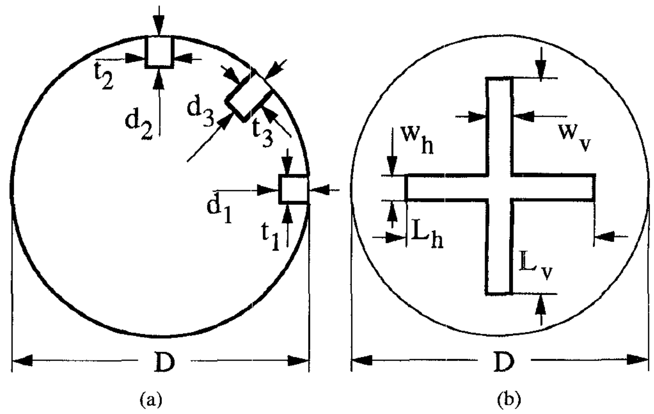

Fig. 2. (a) The "triseptum" circular waveguide and (b) the cross-shaped waveguide considered in the examples. Dimensions: $D = 24$ mm, $t_1 = t_2 = t_3 = 2$ mm, $d_1 = 1.64$ mm, $d_2 = 1.74$ mm, $d_3 = 2.72$ mm, $w_h = w_v = 2$ mm, $L_h = 15.3$ mm, $L_v = 17.3$ mm.

$$I^{\mathrm{TM,TM}}_{pq} = \int_\Omega \vec{\mathcal{E}}''_p \cdot \vec{e}''_q \, ds = k''_p \sum_{n=1}^{N''} R'_{np} b''_n + \frac{a''_p}{k''_p} \tag{8}$$

$$I^{\mathrm{TM,TE}}_{pq} = \int_\Omega \vec{\mathcal{E}}''_p \cdot \vec{e}'_q \, ds = -\frac{1}{\kappa'_q} \sum_{n=1}^{N'} R''_{np} b'_n \tag{9}$$

$$I^{\mathrm{TE,TM}}_{pq} = \int_\Omega \vec{\mathcal{E}}'_p \cdot \vec{e}''_q \, ds = 0 \tag{10}$$

where

$$R_{np} = \frac{1}{k'^2_p} \int_\sigma \vec{\mathcal{E}}'_p \cdot \vec{t} w_n \, d\ell \tag{11}$$

$$R'_{np} = \frac{1}{k''^2_p} \int_\sigma \varphi_p w_n \, d\ell \tag{12}$$

$$R''_{np} = \frac{1}{k''_p} \int_\sigma \varphi_p \frac{\partial w_n}{\partial \ell} \, d\ell \tag{13}$$

and, of course, it is assumed that $p \le M'$ (TE modes) or $p \le M''$ (TM modes). Equation (10) shows that TE modes of the large waveguide never couple with TM modes of the small one, independently of its shape, a result already known in the literature [8].

It is pointed out that coefficients R_{np} and R'_{np} are the entries of matrices $\mathbf{R}$ and $\mathbf{R}'$ that are used for the mode calculation [see [4], eqs. (12d), (16c)]. Therefore, only quantities R''_{np}—which involve simple line integrals—must be computed, all other quantities being already known once the modes of the small waveguide have been determined.

III. Numerical Results

The algorithm has been validated using as benchmarks the step discontinuities between circular/rectangular waveguides, since in these cases the exact values of the coupling coefficients are easily determined. Here, only the results for a junction between two offset rectangular waveguides are reported (dimensions of Ω: $a \times a/2$; dimensions of S: $3a/5 \times a/4$; axes offset: $a/20$ along x and $3a/40$ along y). Running on a SUN SparcStation 10, the code computed in only 11 s the first 10 modes of S and the 1700 coefficients (7)–(9) relative to the first 100 TE and 100 TM modes of Ω (10 s were spent for the numerical determination of the mode spectrum of S

TABLE I
COUPLING COEFFICIENTS AND ABSOLUTE ERRORS (IN PARENTHESIS) FOR THE JUNCTION BETWEEN TWO RECTANGULAR WAVEGUIDES

	TE_{10}	TE_{20}	TE_{01}
TE_{10}	0.63255 ($5.3\,10^{-5}$)	0.04833 ($1.7\,10^{-4}$)	0.00086 ($8.6\,10^{-4}$)
TE_{01}	0.00012 ($1.2\,10^{-4}$)	0.00018 ($1.8\,10^{-4}$)	−0.58468 ($1.2\,10^{-3}$)
TE_{20}	−0.15172 ($3.7\,10^{-4}$)	0.49246 ($3.4\,10^{-4}$)	0.00153 ($1.5\,10^{-3}$)
TE_{11}	−0.16370 ($1.7\,10^{-4}$)	−0.01232 ($2.2\,10^{-4}$)	0.09893 ($5.7\,10^{-4}$)
TM_{11}	0.06190 ($4.0\,10^{-5}$)	−0.20140 ($3.0\,10^{-5}$)	0.28103 ($8.5\,10^{-5}$)
TE_{21}	−0.26393 ($1.1\,10^{-4}$)	−0.25738 ($8.1\,10^{-5}$)	0.00182 ($1.8\,10^{-3}$)
TM_{21}	0.12694 ($5.8\,10^{-5}$)	0.12472 ($8.9\,10^{-4}$)	−0.02402 ($1.2\,10^{-3}$)
TE_{30}	−0.32710 ($4.2\,10^{-5}$)	−0.02505 ($1.8\,10^{-5}$)	−0.04986 ($1.1\,10^{-4}$)
TE_{31}	0.06176 ($9.8\,10^{-5}$)	−0.20126 ($1.7\,10^{-4}$)	−0.28166 ($5.5\,10^{-4}$)
TM_{31}	0.08468 ($8.9\,10^{-5}$)	0.08271 ($1.6\,10^{-4}$)	0.03408 ($1.2\,10^{-4}$)

TABLE II
COUPLING COEFFICIENTS AND CUTOFF WAVENUMBERS (IN [MM^{-1}]) OF THE WAVEGUIDE OF FIG. 2(a)

mode	TE *no.* 1	TE *no.* 2	TM *no.* 1	TE *no.* 4	TE *no.* 4
κ_c	0.150059	0.155937	0.209259	0.246890	0.248848
TE_{11s}	0.69574	0.68396	==	−0.05331	0.03338
TE_{11c}	−0.66897	0.71139	==	0.05855	0.02029
TM_{01}	0.07871	−0.00005	−0.94866	0.06526	−0.00837
TE_{21s}	0.00304	0.00063	==	0.90819	−0.08068
TE_{21c}	0.00067	−0.01091	==	−0.08028	−0.94792

and 1 s for the coupling coefficients). The statistical analysis of the accuracy of the computed coefficients shows that the mean value of the magnitude of the absolute error is 8.5×10^{-4}, that its r.m.s. value is 2.3×10^{-3}, and that the standard deviation is 2.1×10^{-3}. As expected, the accuracy decreases when the order of the modes of the two waveguides increases (the maximum absolute error is 0.03 for the coupling between the TE_{31} mode of S and the TE_{61} mode of Ω). Far better accuracies are obtained for lower-order modes, as shown by the results of Table I, which reports the computed coefficients for the first three modes of S and the first 10 modes of Ω (figures in parentheses are the magnitude of the absolute error). Similar results have been obtained in the other validation cases. Other results are given to prove the efficiency of the algorithm: they concern the two geometries of Fig. 2, which represent the basic discontinuities for modeling circular waveguide dual-mode filters. The dimensions reported in the figure caption are taken from the actual design of a filter operating at about 12 GHz [9]. Running on the said workstation, the code found all the modes of the waveguides of Fig. 2 with cutoff frequencies up to 50 GHz and computed coupling coefficients with all the modes of Ω having cutoff frequencies up to 100 GHz, i.e., 169 TE and 144 TM modes. The computing time to evaluate the mode spectrum of the "triseptum" circular waveguide of Fig. 2(a) was 14 s (45 TE and 31 TM modes were found), and the additional time to compute all the 18 549 coupling coefficients

TABLE III
COUPLING COEFFICIENTS AND CUTOFF WAVENUMBERS (IN [MM^{-1}]) OF THE WAVEGUIDE OF FIG. 2(b)

TE *no.* 1 κ_c=0.18767	TE *no.* 2 κ_c=0.21143	TE *no.* 3 κ_c=0.21309
0.32817 (TE_{11c})	−0.21687 (TE_{21s})	0.31892 (TE_{11s})
−0.37234 (TM_{11s})	−0.04556 (TE_{01})	0.36469 (TM_{11c})
−0.08490 (TE_{31c})	−0.35523 (TM_{21c})	0.06829 (TE_{31s})
0.16012 (TE_{12c})	0.01688 (TE_{41s})	0.20216 (TE_{12s})
0.17080 (TM_{31s})	−0.20726 (TE_{22s})	0.14110 (TM_{31c})

was 7 s. Table II reports the first few coupling coefficients: the modes of S are ordered according to their cutoff wavenumbers, and degenerate modes of Ω are labeled with a "c" (cosine) or "s" (sine) index. Thanks to the symmetry, the code ran even faster in the case of Fig. 2(b). The mode spectrum, computed in 9 s, consists of 9 TE modes. The additional time to compute the coupling coefficients was only 1 s (in this case only 703 coefficients were actually computed, since the coefficients that are zero by symmetry have been automatically skipped): Table III shows the first nonzero coefficients of the first three modes.

IV. CONCLUSION

A fast and accurate algorithm has been presented for the determination of the coupling coefficients between a rectangular or circular waveguide connected to a smaller one of arbitrary shape. The reported examples show that the accuracy is very good, and that the computing time is moderate, even when dealing with a large number of modes in the two waveguides.

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